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The Marginal Pricing Rule Revisited

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Abstract

The purpose of the paper is to introduce a tighter definition for the marginal pricing rule. By means of an example, we illustrate the improvements that one gets with the new definition with respect to the former one with the Clarke's normal cone.

Keywords General economic equilibrium , increasing returns, marginal pricing rule

JEL Classification Numbers: C62, D50, D51

1 Introduction

Guesnerie (1975) is the first who studied the second welfare theorem in a general equilibrium setting with non-convex production sets at the level of generality of Debreu (1959). To modelize the marginal cost pricing rule, he considered the normal cone of Dubovickii and Miljutin, that is, the firm follows the marginal cost pricing rule at a production y for a price vector p if p belongs to the normal cone of Dubovickii and Miljutin of the production set at y. This definition allows to encompass several cases: when the production set is convex, then we recover the standard profit maximizing behavior, when the production set is smooth, the unique normalized price satisfying the marginal cost pricing rule is the unique normalized outward normal vector, and, when the production set is defined by a finite set of smooth inequality constraints satisfying a qualification condition, the normal cone is generated by the gradient vectors of the binding constraints.

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Later, Cornet (1990) (but the first version was written in 1982) proposes to use the Clarke's normal cone (see Clarke (1983)) to represent the marginal pricing rule for the existence problem. Indeed, this cone exhibits two additional fundamental properties: when the production set is closed and satisfies the free-disposal assumption, the Clarke's normal cone is convex and has a closed graph. These properties were used in Bonnisseau-Cornet (1990) to prove the existence of marginal pricing equilibria with several producers.

Then, Khan (1999) (but the first version was written in the eighties) considers the limiting normal cone to extend the second welfare theorem. This cone is not necessarily convex and may be strictly smaller than the Clarke's normal cone. After him, several extensions were made in infinite dimensional spaces. Nevertheless the example of Beato and Mas-Colell (1983) shows that an equilibrium may not exists with the limiting normal cone or with the cone of Dubovickii and Miljutin, although an equilibrium exists with the Clarke's one.

The major drawback of the Clarke's normal cone is that it is too large in the sense that it is defined as the convex hull of the limiting normal cone. So, some vectors belong to the Clarke's normal cone since they are a convex combination of normal vectors but they do not satisfy a "normality" condition. Note also that Jouini (1988) exhibits a production set where the Clarke's normal cone is the positive orthant for every weakly efficient production. So, in that case, the marginal pricing rule puts no restriction on the firm's behavior[1].

The purpose of this note is to introduce a new definition of the marginal pricing rule by considering the so-called intermediate normal cone borrowed from Cornet-Czarnecki (2001). This cone lies between the limiting and the Clarke's normal cones.

After a presentation of the definition of the intermediate normal cone, we provide an example of a production set in a three-commodity economy, which shows that the marginal pricing rule can be not convex valued. This implies that the set of prices satisfying the marginal pricing rule can be strictly smaller than the Clarke's normal cone. The non-convex production set is inspired from an example in Czarnecki-RIfford (2004).

We compute the unique optimal marginal pricing rule equilibrium with our new definition. We also exhibit another price-allocation pair, which is a marginal pricing rule equilibrium when the marginal pricing rule is defined by the Clarke's normal cone. This second equilibrium is not productive efficient. Consequently, the use of the intermediate normal cone to define the marginal pricing rule leads to a tighter notion of equilibria, which may exhibit better optimality properties.

Another advantage of our new definition of the marginal pricing rule is that it comes from an approximation process. So, one can expect a better robustness of the equilibrium set. In our example, we can find a converging sequence of smooth approximation of the non-convex production set such that the equilibrium sets of the approximate smooth economies converge to the unique optimal marginal pricing rule. Hence the sec-

[1] See also the functionnal version [12] and its recent generalizations [3, 4]

ond non efficient marginal pricing rule coming with the Clarke's normal cone disappears with a small perturbation of the production set.

It is interesting to introduce a new notion of normal cone if we can expect that a marginal pricing equilibrium exists under reasonable assumptions like the ones of Bonnisseau-Cornet (1990). Even if the marginal pricing rule is not convex valued, the intermediate normal cone enjoys fixed-point like properties, which allows us to conjecture that we can get the existence of an equilibrium under rather general conditions.

It is worth to note that our new definition does not answer all criticisms. Indeed, for the Jouini's example, the marginal pricing rule with the intermediate normal cone is again the whole positive orthant. But, we illustrate another open question with our example. Indeed, Cornet-Czarnecki (2001) consider external smooth approximation of the sets. It is also possible for the production sets to consider internal smooth approximation since they are epi-lipschitz. We show that the intermediate normal cone is not the same when we consider external or internal approximations. At this stage, we do not know what is the right approximation process leading to the smallest possible normal cone.

2 The marginal pricing rule

We[2] consider a production set Y in a ℓ-commodity economy, that is Y is a subset of $\mathbb{R}^\ell$ satisfying the following assumption:

Assumption P: Y is a nonempty, closed subset of $\mathbb{R}^\ell$, $Y \neq \mathbb{R}^\ell$ and $Y - \mathbb{R}^\ell_+ = Y$.

We now recall several definitions of normal vectors and normal cones. The proximal normal cone to Y at $y \in Y$ is the set of the perpendicular vectors to Y at y, that is

$$N_Y^P(y) = \{p \in \mathbb{R}^\ell \mid \exists \rho > 0, B(y + \rho p, \rho\|p\|) \cap Y = \emptyset\}$$

The limiting normal cone and the Clarke's normal cone to Y at y are then defined by:

$$N_Y^L(y) = \limsup_{y' \in Y, y' \to y} N_Y^P(y')$$

where lim sup is taken in the sense of Painlevé, which means that $p \in N_Y^L(y)$ if there exists a sequence (y^ν, p^ν) converging to (y, p) and such that $y^\nu \in Y$ and $p^\nu \in N_Y^P(y^\nu)$ for every integers ν.

$$N_Y^C(y) = \mathrm{clco} N_Y^L(y)$$

To introduce the new notion of normal cone, we consider the distance function d_Y to Y and its generalized gradient. We know that d_Y is Lipschitz and, thus, from

[2]Notations: if p and y are vectors of $\mathbb{R}^\ell$, $p \cdot y$ is the usual inner product of $\mathbb{R}^\ell$ and $\|y\|$ is the associated norm. For $\rho > 0$, $B(y, \rho)$ is the open ball of center y and radius ρ. If Y is a subset of $\mathbb{R}^\ell$, then cl$Y$ (resp. co$Y$, bd$Y$, cone$Y$) denotes the closure (resp. the convex hull, the boundary, the convex conic hull) of Y. d_Y is the distance function to Y, that is $d_Y(y) = \inf\{\|y - y'\| \mid y' \in Y\}$.

Rademacher's Theorem, almost everywhere differentiable. We denote by $\mathrm{dom}(\nabla d_Y)$ the domain on which d_Y is differentiable. The Clarke's generalized gradient $\partial d_Y(y)$ of d_Y at y is defined as:

$$\partial d_Y(y) = \mathrm{co} \limsup_{y' \in \mathrm{dom}(\nabla d_Y), y' \to y} \nabla d_Y(y')$$

For $y \in Y$, the intermediate normal cone is defined as follows:

$$N_Y^I(y) = \cup_{t \geq 0} t \limsup_{y' \notin Y, y' \to y} \partial d_Y(y')$$

We recall the following elementary properties of the above normal cones. The proof can be found in Cornet-Czarnecki (2001).

Proposition 1 *For every $y \in Y$,*

a) $N_Y^L(y) \subset N_Y^I(y) \subset N_Y^C(y)$;

b) Under Assumption P, the three normal cones are included in $\mathbb{R}^\ell_+$ and are different from $\{0\}$.

c) If Y is convex, the three normal cones coincide with the usual normal cone of the convex analysis, that is $\{p \in \mathbb{R}^\ell \mid p \cdot y \geq p \cdot y', \forall y' \in Y\}$.

We propose to define the marginal pricing rule, which associates to a weakly efficient production the set of admissible prices as follows.

Definition 1 The marginal pricing rule *A price-production pair $(p, y) \in \mathbb{R}^\ell \setminus \{0\} \times \mathrm{bd}Y_j$ satisfies the marginal pricing rule if:*

$$p \in N_Y^I(y).$$

The interest of this new definition with respect to the previous one using the Clarke's normal cone is that, for a given production, it provides always a smaller set of prices satisfying the marginal pricing rule since the intermediate normal cone is included in the Clarke's normal cone. But our main interest is the existence problem of a marginal pricing equilibrium, since for the second welfare theorem, the limiting normal cone, which is the smaller one, is sufficient.

The intermediate normal cone is not always convex valued, as it is illustrated in the example below. But, we could expect getting an equilibrium through an approximation argument. Indeed, in Cornet-Czarnecki (2001), a smooth approximation result is proved for the compact epi-Lipschitzian sets. Actually, the result is true for a larger class but we restrict ourself to the epi-Lipschitzian ones since Assumption P implies that Y is epi-Lipschitzian. In the following, $G(N_Y^C)$ (resp. $G(N_Y^I)$) denotes the graph of the Clarke's (resp. intermediate) normal cone, that is

$$G(N_Y^C) = \{(y, p) \in Y \times \mathbb{R}^\ell \mid p \in N_Y^C(y)\}$$

Theorem 2.1 *Let Y be a compact epi-Lipschitizian subset of $\mathbb{R}^\ell$. Y admits a smooth normal approximation $(Y_k)_{k\in\mathbb{N}}$ in the sense that:*

(i) for every k, Y_k is a compact and smooth subset of $\mathbb{R}^\ell$, that is a closed C^∞ submanifold with boundary of $\mathbb{R}^\ell$ of full dimension;

(ii) for every k, $Y_{k+1} \subset Y_k \subset \{y \in \mathbb{R}^\ell \mid d_Y(y) < 1\}$ and $Y = \cap_{k\in\mathbb{N}} Y_k$;

(iii) $\limsup_{k\to\infty} G(N^C_{Y_k}) \subset G(N^I_Y)$.

Note that the three normal cones to Y_k coincide since Y_k is a smooth sub-manifold of $\mathbb{R}^\ell$. The problem to apply this result for the existence of an equilibrium comes from the fact that Assumption P implies that Y is not bounded, hence non-compact. So the approximation argument cannot be directly done on the production set but on a compact subset of it.

In the next section, we provide an example of a three-commodity economy with two producers. We show that a marginal pricing equilibrium exists for the marginal pricing rule defined by the intermediate normal cone and a non-efficient additional price-allocation pair exists where the price belongs to the Clarke's normal cones at the production. So, the use of the intermediate normal cone allows us to provide a more precise result on the set of equilibria.

3 An Example

We consider a three-good economy with one consumer and two producers. The utility function of the unique consumer is $u(a, b, c) = abc$ and his initial endowments is $\omega = (1, 3, 1)$. The two first commodities are inputs and the third one is an output. The production sets are defined by mean of production functions.

$$Y_1 = \{(a, b, c) \in \mathbb{R}^3 \mid a \leq 0, b \leq 0, c \leq \max\{-a, -b\}\}$$
$$Y_2 = \{(a, b, c) \in \mathbb{R}^3 \mid a \leq 0, b \leq 0, c \leq -(3/2)a - (1/2)b\}$$

3.1 Computation of the marginal pricing rule

The second producer has a convex production technology with constant returns. So, the marginal pricing rule coincides with the maximization of profit, and the normal cone is the standard normal cone of the convex analysis.

For the first producer, the technology is non convex even if there are constant returns to scale. We now describe the normal cones for the weakly efficient productions in Y_1. We have nine different cases but the most interesting one is at the origin. Except on the half line $\{(a, b, c) \in \mathbb{R}^3 \mid a = b \leq 0, c = -a\}$, the three definitions of the normal cone coincide and the computation is easy since the production set is a polyedral convex set in a neighborhood of the production.

Case I: $y_1 = (a,b,c)$, $a < 0, b = 0, c < -a$, $N^L_{Y_1}(y_1) = N^I_{Y_1}(y_1) = N^C_{Y_1}(y_1) = \text{cone}\{(0,1,0)\}$;

Case II: $y_1 = (a,b,c)$, $a = b = 0, c < 0$,
$N^L_{Y_1}(y_1) = N^I_{Y_1}(y_1) = N^C_{Y_1}(y_1) = \text{cone}\{(1,0,0),(0,1,0)\}$;

Case III: $y_1 = (a,b,c)$, $a = 0, b < 0, c < -b$, $N^L_{Y_1}(y_1) = N^I_{Y_1}(y_1) = N^C_{Y_1}(y_1) = \text{cone}\{(1,0,0)\}$,

Case IV: $y_1 = (a,b,c)$, $a = 0, b < 0, c = -b$, $N^L_{Y_1}(y_1) = N^I_{Y_1}(y_1) = N^C_{Y_1}(y_1) = \text{cone}\{(1,0,0),(0,1,1)\}$;

Case V: $y_1 = (a,b,c)$, $b < a < 0, c = -b$, $N^L_{Y_1}(y_1) = N^I_{Y_1}(y_1) = N^C_{Y_1}(y_1) = \text{cone}\{(0,1,1)\}$;

Case VI: $y_1 = (a,b,c)$, $a = b < 0, c = -b$, $N^L_{Y_1}(y_1) = \text{cone}\{(1,0,1)\} \cup \text{cone}\{(0,1,1)\}$;
$N^I_{Y_1}(y_1) = N^C_{Y_1}(y_1) = \text{cone}\{(1,0,1),(0,1,1)\}$;

Case VII: $y_1 = (a,b,c)$, $a < b < 0, c = -a$, $N^L_{Y_1}(y_1) = N^I_{Y_1}(y_1) = N^C_{Y_1}(y_1) = \text{cone}\{(1,0,1)\}$;

Case VIII: $y_1 = (a,b,c)$, $a < 0, b = 0, c = -a$,
$N^L_{Y_1}(y_1) = N^I_{Y_1}(y_1) = N^C_{Y_1}(y_1) = \text{cone}\{(1,0,1),(0,1,0)\}$;

Case IX: $y_1 = (0,0,0)$,
$N^L_{Y_1}(y_1) = \text{cone}\{(1,0,0),(0,1,0),(1,1,1)\} \cup \text{cone}\{(1,0,1),(1,1,1)\} \cup$
$\text{cone}\{(0,1,1),(1,1,1)\}$;
$N^I_{Y_1}(y_1) = \text{cone}\{(1,0,0),(0,1,0),(1,1,1)\} \cup \text{cone}\{(1,0,1),(0,1,1),(1,1,1)\}$;
$N^C_{Y_1}(y_1) = \text{cone}\{(1,0,0),(0,1,0),(1,0,1),(0,1,1)\}$;

We remark that at the production $(0,0,0)$ the limiting normal cone is strictly included in the intermediate normal cone, which is strictly included in the Clarke's normal cone, and is not convex.

To obtain the above formula, it suffices to study the distance function d_{Y_1} to Y_1. Since Y_1 is the union of two closed convex cones $Z_1 = \{(a,b,c) \in \mathbb{R}^3 \mid a \leq 0, b \leq 0, c \leq -a\}$ and $Z_2 = \{(a,b,c) \in \mathbb{R}^3 \mid a \leq 0, b \leq 0, c \leq -b\}$, d_{Y_1} is the infimum of d_{Z_1} and d_{Z_2}. The functions d_{Z_1} and d_{Z_2} are convex and continuously differentiable outside Y_1. Consequently, the generalized gradient of d_{Y_1} at y_1 outside Y_1 is reduced to the gradient of d_{Z_1} or d_{Z_2} at y_1 if $d_{Z_1}(y_1)$ is strictly less than $d_{Z_2}(y_1)$ or $d_{Z_2}(y_1)$ is strictly less than $d_{Z_1}(y_1)$. When $d_{Z_1}(y_1)$ is equal to $d_{Z_2}(y_1)$, then the generalized gradient of d_{Y_1} is the convex hull of the gradients of d_{Z_1} and d_{Z_2}. Consequently, the gradient of d_{Y_1} is not reduced to a singleton only if the two gradients of d_{Z_1} and d_{Z_2} differ, that is when the projections of y_1 on Z_1 and Z_2 are not the same. This holds true on the following domains:

$$D_1 = \{(a,b,c) \in \mathbb{R}^3 \mid a = b \geq 0, c > a\}$$

$$D_2 = \{(a,b,c) \in \mathbb{R}^3 \mid a = b < 0, c > -a\}$$

If $y_1 = (a,a,c) \in D_1$,
the projection on Z_1 is $(\frac{a-c}{2}, 0, \frac{c-a}{2})$ and the gradient of d_{Z_1} is $\frac{1}{\sqrt{\frac{(a+c)^2}{2}+a^2}}(\frac{a+c}{2}, a, \frac{a+c}{2})$,
the projection on Z_2 is $(0, \frac{a-c}{2}, \frac{c-a}{2})$ and the gradient of d_{Z_2} is $\frac{1}{\sqrt{\frac{(a+c)^2}{2}+a^2}}(a, \frac{a+c}{2}, \frac{a+c}{2})$,
and $\partial d_{Y_1}(y_1) = \frac{1}{\sqrt{\frac{(a+c)^2}{2}+a^2}}\mathrm{co}\{(\frac{a+c}{2}, a, \frac{a+c}{2}), (a, \frac{a+c}{2}, \frac{a+c}{2})\}$.

If $y_1 = (a,a,c) \in D_2$,
the projection on Z_1 is $(\frac{a-c}{2}, a, \frac{c-a}{2})$ and the gradient of d_{Z_1} is $\frac{\sqrt{2}}{a+c}(\frac{a+c}{2}, 0, \frac{a+c}{2})$,
the projection on Z_2 is $(a, \frac{a-c}{2}, \frac{c-a}{2})$ and the gradient of d_{Z_2} is $\frac{\sqrt{2}}{a+c}(0, \frac{a+c}{2}, \frac{a+c}{2})$,
and $\partial d_{Y_1}(y_1) = \frac{\sqrt{2}}{a+c}\mathrm{co}\{(\frac{a+c}{2}, 0, \frac{a+c}{2}), (0, \frac{a+c}{2}, \frac{a+c}{2})\}$.

3.2 Computation of Equilibria

A marginal pricing equilibrium is a collection (p^*, x^*, y_1^*, y_2^*) in $\mathbb{R}^3_+ \setminus \{0\} \times \mathbb{R}^3_+ \times \mathrm{bd}Y_1 \times \mathrm{bd}Y_2$ such that $x^* = y_1^* + y_2^* + \omega$, $p^* \in N^I_{Y_1}(y_1^*) \cap N^I_{Y_2}(y_2^*)$ and x^* is a solution of the maximization of the utility function u on the budget set $\{x \in \mathbb{R}^3_+ \mid p^* \cdot x \leq p^* \cdot (y_1^* + y_2^* + \omega)\}$.

Using the definition of the marginal pricing rule by the intermediate normal cone, the economy admits only one marginal pricing equilibrium, which is

$$\left(q^* = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right), \xi^* = (1,2,2), \zeta_1^* = (0,-1,1), \zeta_2^* = (0,0,0)\right)$$

To check that this is the only one equilibrium, we can remark that the equilibrium price must belong to $\mathrm{cone}\{(1,0,0), (0,1,0), (\frac{3}{7}, \frac{2}{7}, \frac{2}{7})\}$. Furthermore, the equilibrium is also an equilibrium of the economy with the global production set $Y = Y_1 + Y_2$, which is convex. Actually $Y = \{(a,b,c) \in \mathbb{R}^3 \mid c \leq -(3/2)a - b\}$. So we have just to compute a competitive equilibrium with a unique producer having constant return to scale. This equilibrium is optimal since it satisfies the necessary and sufficient conditions of optimality for the global production set Y.

The new definition of the marginal pricing rule allows us to withdraw a non-efficient equilibrium that appears with the Clarke's normal cone. This is a direct consequence of the fact that the intermediate normal cone at 0 is strictly smaller than the Clarke's normal cone.

One easily checks that

$$\left(p^* = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right), x^* = \left(\frac{8}{9}, \frac{8}{3}, \frac{4}{3}\right), y_1^* = (0,0,0), y_2^* = \left(-\frac{1}{9}, -\frac{1}{3}, \frac{1}{3}\right)\right)$$

satisfies the definition of a marginal pricing equilibrium if we replace the intermediate normal cone by the Clarke's normal cone. Note that the consumer strictly prefers ξ^* to x^* and the productions (y_1^*, y_2^*) are not productive efficient. Indeed, the quantity of the output is $\frac{1}{3}$ whereas with the quantities of inputs $(-\frac{1}{9}, -\frac{1}{3})$, one can produce $\frac{1}{2}$ if the quantity of the first commodity is given to the second producer and the quantity of the second commodity is given to the first producer.

4 Open Question

The intermediate normal cone leads to a tighter marginal pricing rule. But it remains to know if it is the tightest or not. The following computation shows that the answer is no since by considering another definition of the normal cone through an interior approximation, we get a different normal cone. In our example, the primary definition gives the smallest normal cone. But, if we reverse the production set as it is explained below, the contrary holds true. Thus, we have provided an improvement by introducing this new definition of the marginal pricing rule, but the question to define the "best" normal cone compatible with the existence of an equilibrium is still open. Note that the question is irrelevant with the Clarke's normal cone since for the epi-lipschitzian set, it is known that the normal cone coincides with the opposite of the normal cone of the closure of the complementary.

We compute now the intermediate normal cone by an interior approximation. This means that we consider Γ_1, the opposite of the closure of the complementary of Y_1. An exterior approximation of Γ_1 is an interior approximation of Y_1.

We can find a difference between $N^I_{Y_1}(y_1)$ and $N^I_{\Gamma_1}(-y_1)$ only at $(0,0,0)$ since, otherwise the set Y_1 is either an half space, the intersection of two half spaces or the union of two half spaces, locally around y_1. Indeed, the following shows that $N^I_{\Gamma_1}(0,0,0)$ is the union of the three following cones: $\text{cone}\{(1,0,0),(0,1,0),(\frac{1}{2},\frac{1}{2},1)\}$, $\text{cone}\{(1,0,0),(0,\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2},1)\}$ and $\text{cone}\{(0,1,0),(\frac{1}{2},0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2},1)\}$.

Thus, $N^I_{\Gamma_1}(0,0,0)$ is strictly than $N^I_{Y_1}(0,0,0)$ since the interior of the two cones $\text{cone}\{(1,0,0),(\frac{2}{5},\frac{1}{5},\frac{2}{5}),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$ and $\text{cone}\{(0,1,0),(\frac{1}{5},\frac{2}{5},\frac{2}{5}),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$ are included in $N^I_{\Gamma_1}(0,0,0)$ and are not included in $N^I_{Y_1}(0,0,0)$. Nevertheless, $N^I_{\Gamma_1}(0,0,0)$ is still strictly smaller than the Clarke's normal cone and non-convex.

We now compute the generalized gradient of the distance function d_{Γ_1} outside Γ_1. We remark that Γ_1 is the union of the three following convex cones: $H_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x \leq 0\}$, $H_2 = \{(x,y,z) \in \mathbb{R}^3 \mid y \leq 0\}$ and $H_3 = \{(x,y,z) \in \mathbb{R}^3, x+z \leq 0, y+z \leq 0\}$. For $(x,y,z) \notin \Gamma_1$, let $K(x,y,z) = \{k \in \{1,2,3\} \mid d_{H_k}(x,y,z) = d_{\Gamma_1}(x,y,z)\}$.

The distance function d_{H_k} are defined as follows:

$d_{H_1}(x,y,z) = x$ and $\nabla d_{H_1}(x,y,z) = (1,0,0)$;

$d_{H_2}(x,y,z) = y$ and $\nabla d_{H_1}(x,y,z) = (0,1,0)$;

$d_{H_3}(x,y,z)$ is equal to:

$$\begin{cases} -(x+z)\frac{\sqrt{2}}{2} & \text{if } -x+2y+z \leq 0 \\ \frac{1}{3}\sqrt{(2x-y+z)^2+(-x+2y+z)^2+(x+y+2z)^2} & \text{if } \begin{cases} -x+2y+z > 0 \\ 2x-y+z > 0 \end{cases} \\ -(y+z)\frac{\sqrt{2}}{2} & \text{if } 2x-y+z \leq 0 \end{cases}$$

$$\text{and } \nabla d_{H_1}(x,y,z) = \begin{cases} (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}) & \text{if } -x+2y+z \leq 0 \\ \frac{(2x-y+z, -x+2y+z, x+y+2z)}{\sqrt{(2x-y+z)^2+(-x+2y+z)^2+(x+y+2z)^2}} & \text{if } \begin{cases} -x+2y+z > 0 \\ 2x-y+z > 0 \end{cases} \\ (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) & \text{if } 2x-y+z \leq 0 \end{cases}$$

So, if (x,y,z) has a unique projection and Γ_1, which happens when $K(x,y,z)$ is a singleton, the generalized gradient of the distance function to Γ_1 is a singleton in the set $\{(1,0,0),(0,1,0)\} \cup \{\frac{\sqrt{2}}{2\sqrt{t^2+1-t}}(t, 1-t, 1) \mid t \in [0,1]\}$.

We now study the case where $K(x,y,z)$ is not a singleton. Then, $\partial d_{\Gamma_1}(x,y,z) = \text{co}\{\nabla d_{H_k}(x,y,z) \mid k \in K(x,y,z)\}$. $K(x,y,z) = \{1,2,3\}$ if $y = x$ and $z = (\sqrt{3}-1)x$. Then

$$\partial d_{\Gamma_1}(x,y,z) = \text{co}\{(1,0,0),(0,1,0), \frac{\sqrt{2}}{\sqrt{3}}(\frac{1}{2}, \frac{1}{2}, 1)\}$$

Considering the limit when x tends to 0, one concludes that

$$\text{cone}\{(1,0,0),(0,1,0),(\frac{1}{2}, \frac{1}{2}, 1)\} \subset N^I_{\Gamma_1}(0,0,0)$$

$K(x,y,z) = \{1,2\}$ if $y = x$ and $z > (\sqrt{3}-1)x$. Then

$$\partial d_{\Gamma_1}(x,y,z) = \text{co}\{(1,0,0),(0,1,0)\}$$

With respect to the previous case, this does not add new elements in the intermediate normal cone.

$K(x,y,z) = \{1,2\}$ if there exists $\alpha > 0$ and $\lambda \in [0, \frac{1}{2}]$ such that $(x,y,z) = \alpha(\sqrt{2}r, \sqrt{2}r + 1 - 2\lambda, -\sqrt{2}r + \lambda + 1)$, with $r = \sqrt{\lambda^2 - \lambda + 1}$. Then,

$$\partial d_{\Gamma_1}(x,y,z) = \text{co}\{(1,0,0), \frac{1}{r\sqrt{2}}(\lambda, 1-\lambda, 1)$$

Considering the limit when α tends to 0, one concludes that

$$\text{cone}\{(1,0,0),(0,\frac{1}{2}, \frac{1}{2}),(\frac{1}{2}, \frac{1}{2}, 1)\} \subset N^I_{\Gamma_1}(0,0,0)$$

$K(x,y,z) = \{1,2\}$ if there exists $\alpha > 0$ and $\lambda \in [\frac{1}{2}, 1]$, such that $(x,y,z) = \alpha(\sqrt{2}r + 2\lambda - 1, \sqrt{2}r, -\sqrt{2}r + 2 - \lambda)$, with $r = \sqrt{\lambda^2 - \lambda + 1}$. Then,

$$\partial d_{\Gamma_1}(x,y,z) = \text{co}\{(0,1,0), \frac{1}{r\sqrt{2}}(\lambda, 1-\lambda, 1)$$

Considering the limit when α tends to 0, one concludes that

$$\text{cone}\{(0,1,0),(\frac{1}{2}, 0, \frac{1}{2}),(\frac{1}{2}, \frac{1}{2}, 1)\} \subset N^I_{\Gamma_1}(0,0,0)$$

Combining all the possibilities, leads to the result.

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